Starter Questions

Find the gradient of the tangent to the curve
$y = x^3 - 2x^2 + 2x - 1$ at the point (-1,-6)

Find the equation of the tangent to the curve
$y = 5 - 10x + x^3$ at the point when x = -1

Find the values of x for which the tangents to the curve
$y = 3x^3 + 6x^2 - 2x + 5$ are parallel to the graph
y - 3x = 2

Starter Questions

Find the gradient of the tangent to the curve $y = x^3 - 2x^2 + 2x - 1$ at the point (-1,-6)

$$\frac{dy}{dx} = 3x^2 - 4x + 2$$

$$gradient = 3 \times (-1)^2 - 4 \times (-1) + 2$$
$$= 9$$

Starter Questions

Find the equation of the tangent to the curve
$y = 5 - 10x + x^3$ at the point when x = -1

$$\frac{dy}{dx} = -10 + 3x^2 \qquad Gradient = -10 + 3 \times (-1)^2$$

$$= -7$$

$$x = -1 \quad y = 5 - 10 \times (-1) + (-1)^3$$

$$Equation\ of\ tangent\ (y - 14) = -7(x + 1)$$

$$y + 7x - 7 = 0$$

Starter Questions

Find the values of x for which the tangents to the curve $y = 3x^3 + 6x^2 - 2x + 5$ are parallel to the graph
y - 3x = 2

$$\frac{dy}{dx} = 9x^2 + 12x - 2$$

$$y = 3x + 2 \quad Gradient = 3$$

$$9x^2 + 12x - 2 = 3$$

$$9x^2 + 12x - 5 = 0$$

$$(3x - 1)(3x + 5) = 0 \quad x = \frac{1}{3} \quad x = -\frac{5}{3}$$

G Differentiation

G1

Understand and use the derivative of $f(x)$ as the gradient of the tangent to the graph of $y = f(x)$ at a general point (x, y); the gradient of the tangent as a limit; interpretation as a rate of change; sketching the gradient function for a given curve; second derivatives; differentiation from first principles for small positive integer powers of x and for $\sin x$ and $\cos x$

Understand and use the second derivative as the rate of change of gradient; connection to convex and concave sections of curves and points of inflection.

G Differentiation

G3	Apply differentiation to find gradients, tangents and normals, maxima and minima and stationary points, points of inflection. Identify where functions are increasing or decreasing.

https://sites.google.com/view/tlmaths/home/a-level-maths/as-only/g-differentiation/g3-gradients#h.aj72ijsk10be

G3-07, G3-08, G3-11, G3-13, G3-14

G Differentiation

Students should:

- understand and be able to use the fact that at a stationary point, $\dfrac{dy}{dx} = 0$
- describe a stationary point as a (local) maximum or minimum
- know that:

At a maximum $\dfrac{dy}{dx}$ = 0 and $\dfrac{d^2y}{dx^2} < 0$

At a minimum $\dfrac{dy}{dx}$ = 0 and $\dfrac{d^2y}{dx^2} > 0$

Note:

the case $\dfrac{d^2y}{dx^2} = 0$ will not be tested at AS

- use $m_1 \times m_2 = -1$ for gradients of tangent and normal
- be able to answer questions set in the form of a practical problem where a function of a single variable has to be optimised
- be able to show that a function is increasing or decreasing, by showing $\dfrac{dy}{dx} > 0$ or $\dfrac{dy}{dx} < 0$ respectively.

4.5 The Second Derivative

We saw last lesson that velocity was the rate of change of displacement over time:

We also saw that acceleration was the rate of change of velocity over time.

This means that acceleration is the second derivative of displacement.

4.5 The Second Derivative

The second derivative is obtained by differentiating a function twice.

e.g.

We can also use the notation: , and .

4.5 The Second Derivative

Example

Given that $y = 3x^5 + \frac{4}{x^2}$ find:

a $\frac{dy}{dx}$

b $\frac{d^2y}{dx^2}$

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4.5 The Second Derivative

Example

Given that $f(x) = 3\sqrt{x} + \dfrac{1}{2\sqrt{x}}$, find:

a $f'(x)$ **b** $f''(x)$

Tip: Easier to differentiate again if you leave it in index form.

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4.5 The Second Derivative

Exercise 3.1

Q1 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of these functions:

a) $y = x^3$ b) $y = x^5$ c) $y = x^4$ d) $y = x$

e) $y = \frac{1}{x}$ f) $y = \sqrt{x}$ g) $y = \frac{1}{x^2}$ h) $y = x\sqrt{x}$

Q2 Find $f'(x)$ and $f''(x)$ for each of these functions:

a) $f(x) = x(4x^2 - x)$ b) $f(x) = (x^2 - 3)(x - 4)$

c) $f(x) = \frac{4x^5 + 12x^3 - 40x}{4(x^2 + 5)}$ d) $f(x) = 3\sqrt{x} + x\sqrt{x}$

e) $f(x) = \frac{1}{x}(3x^4 - 2x^3)$ f) $f(x) = \frac{x^2 - x\sqrt{x} + 7x}{\sqrt{x}}$

Q3 Find the value of the second derivative at the given value for x.

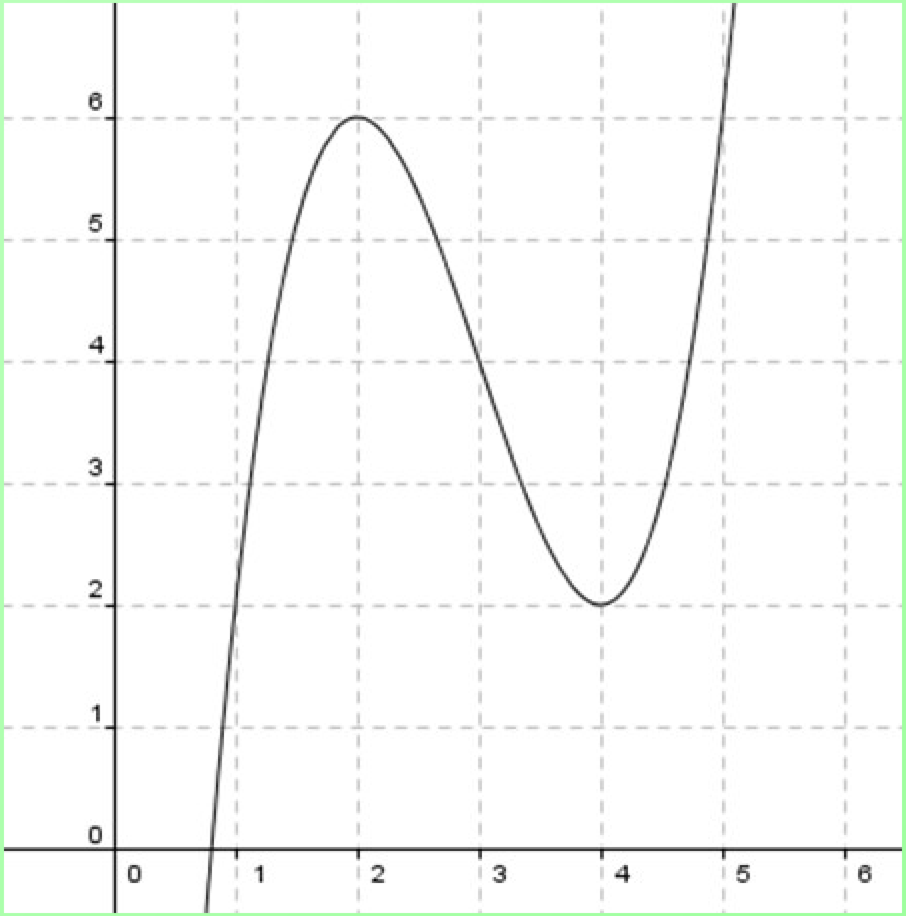
a) $f(x) = x^3 - x^2, \ x = 3$ b) $y = x\sqrt{x} - \frac{1}{x}, \ x = 4$

c) $f(x) = x^2(x - 5)(x^2 + x), \ x = -1$ d) $y = \frac{x^5 + 4x^4 - 12x^3}{x + 6}, \ x = 5$

e) $f(x) = \frac{9x^2 + 3x}{3\sqrt{x}}, \ x = 1$ f) $y = \left(\frac{1}{x^2} + \frac{1}{x}\right)(5 - x), \ x = -3$

4.5 Turning Points

Turning points are where the curve changes from increasing to decreasing (maximum) or vice versa (minimum).



e.g. this function has a maximum at (2, 6) and a minimum at (4, 2).

These are also known as **stationary points**.

At a turning point, .

4.5 Turning Points

Example 3

Work out the coordinates of the turning point on the curve .

At a turning point,

there is a turning point at (-2, -16).

Compare with finding the vertex by completing the square!

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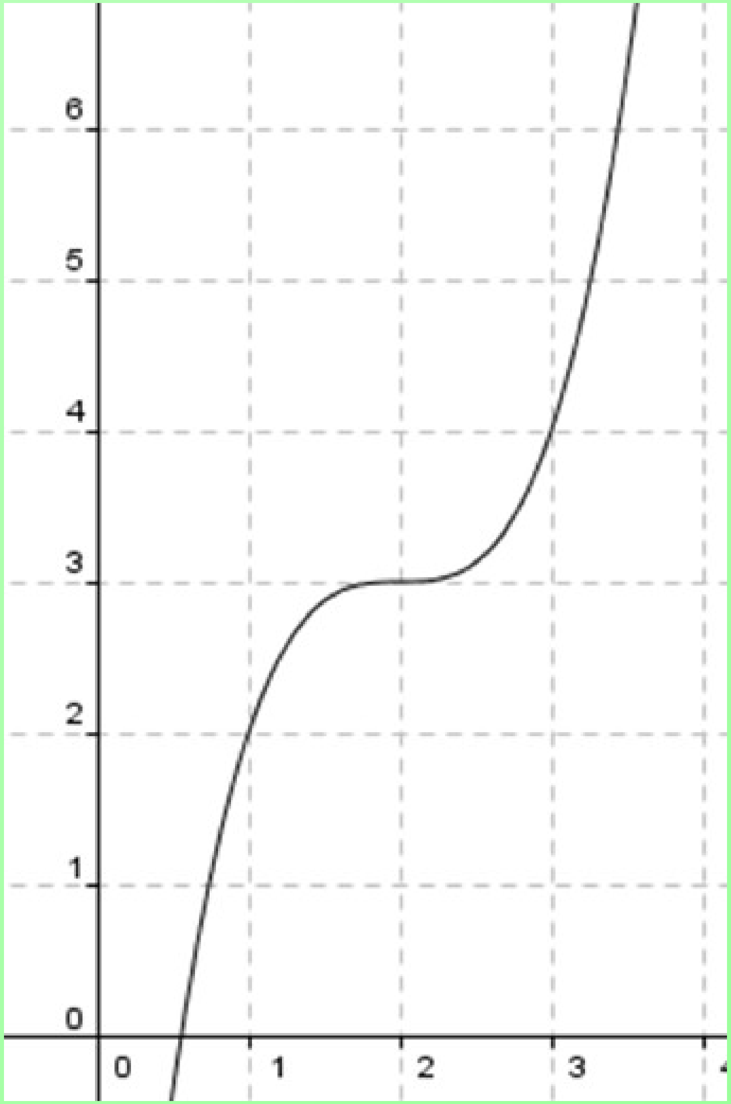
4.5 Turning Points

Turning points are either a local maximum or a local minimum.

However, there are three types of stationary point:
1. Local maximum
2. Local minimum
3. Point of inflection

e.g. this curve has a point of inflection at (2, 3).



You will learn more about

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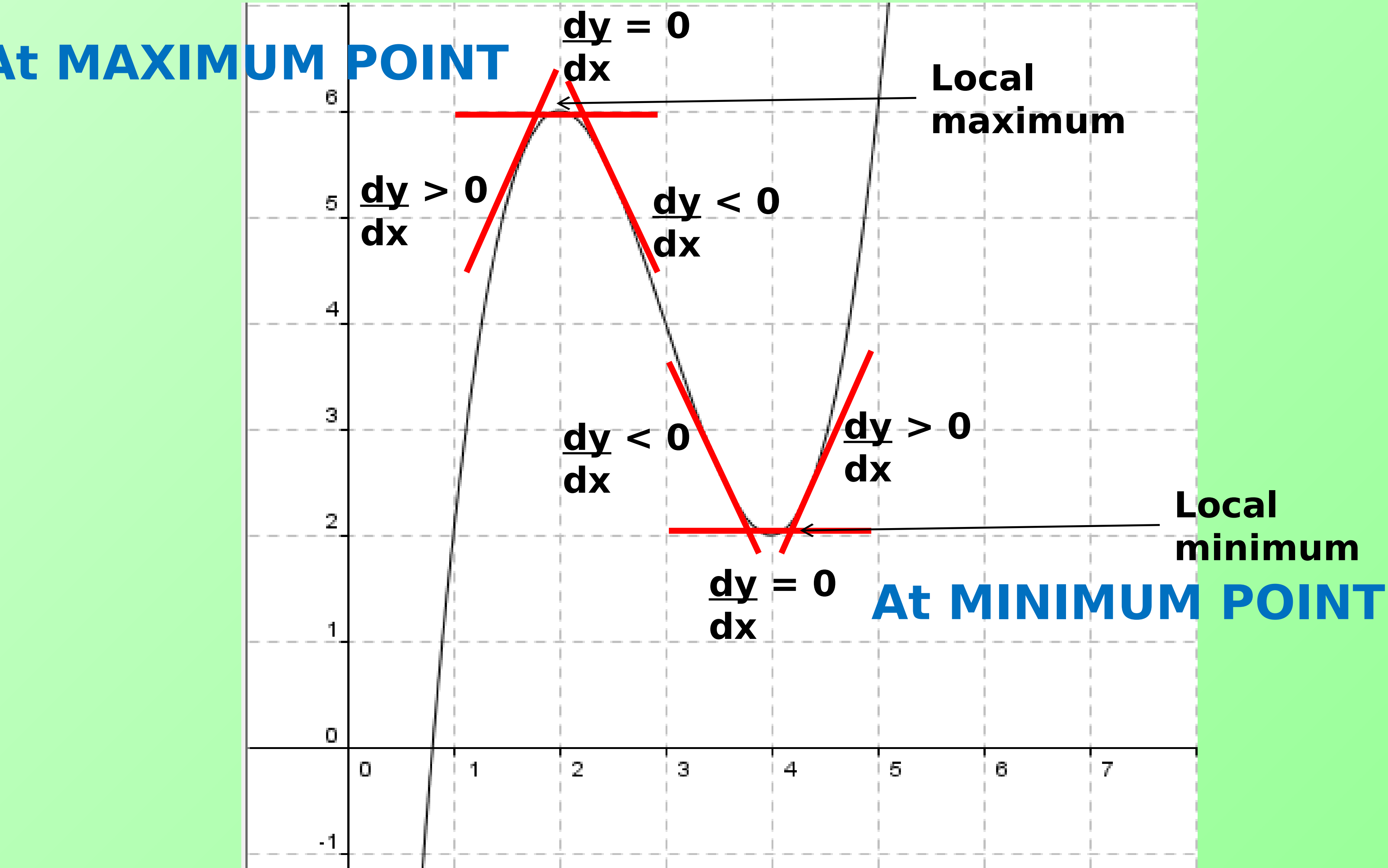
4.5 Turning Points



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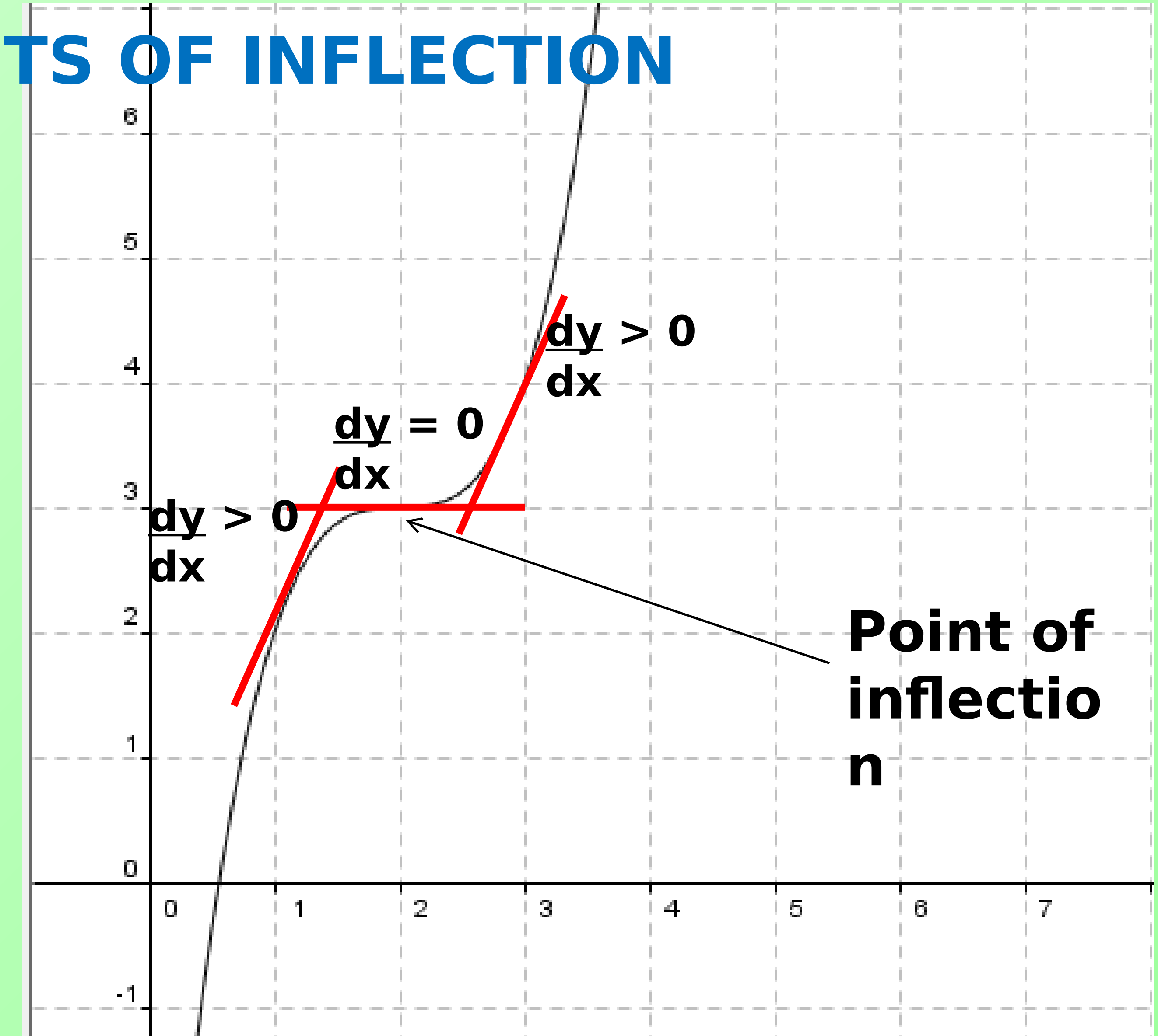
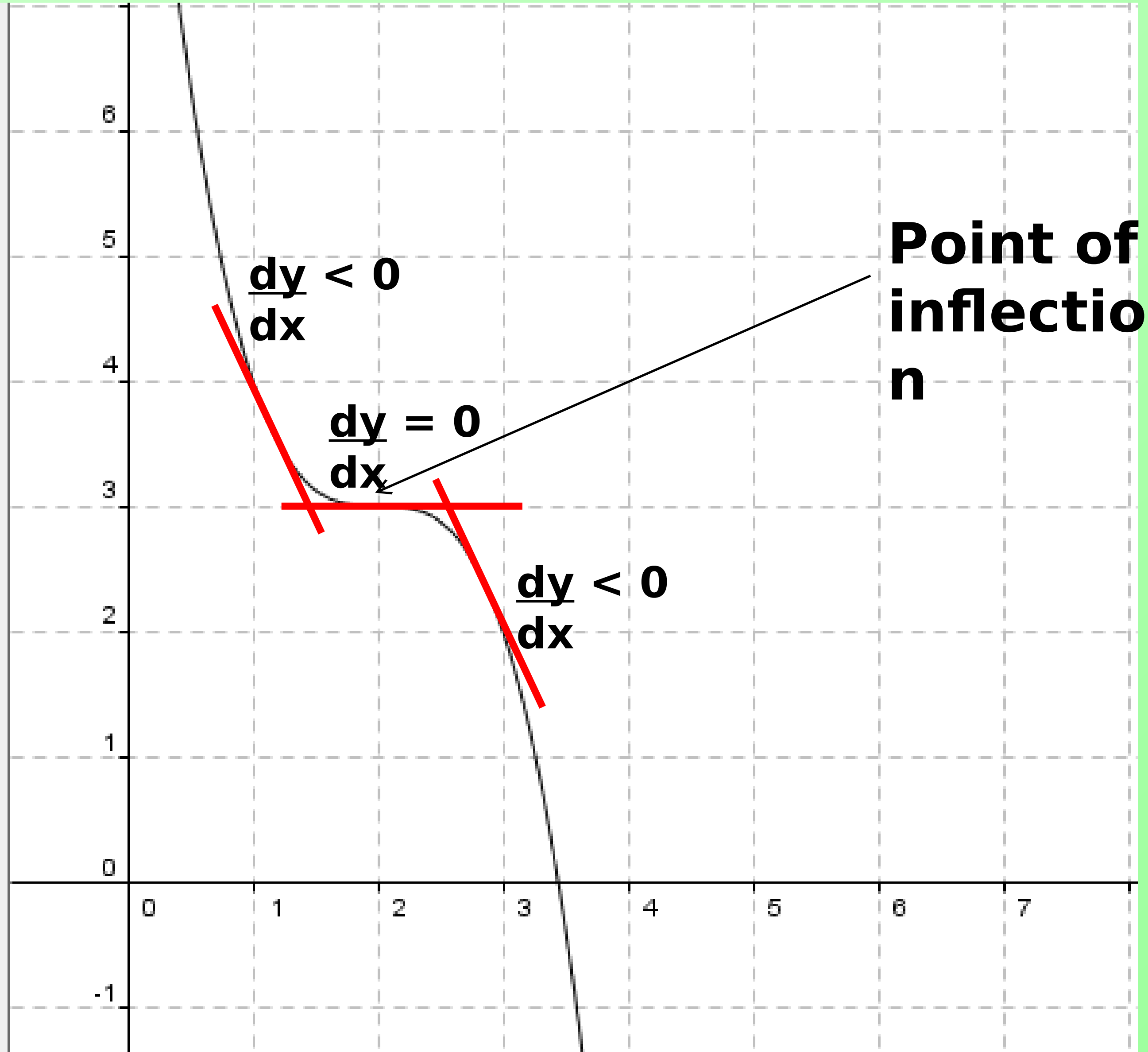
4.5 Turning Points

At POINTS OF INFLECTION



4.5 Turning Points



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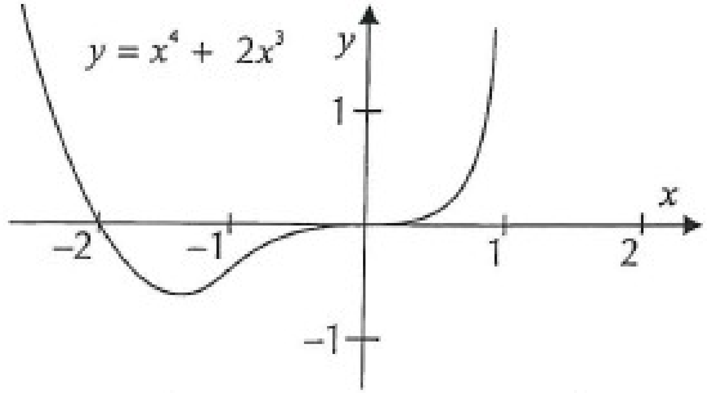
4.5 Turning Points

Exercise 3.2

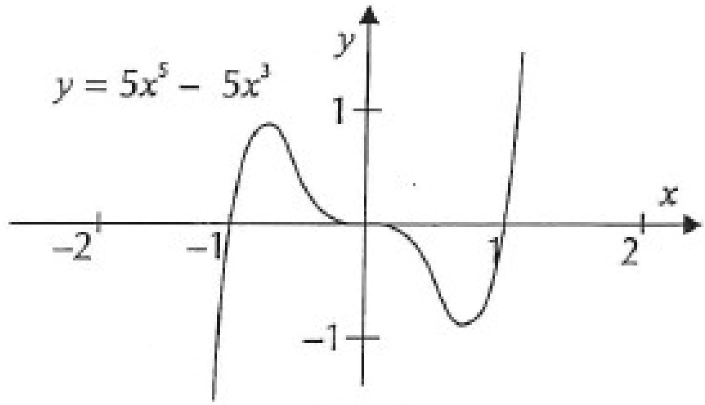
Q1 Without doing any calculations, say how many stationary points the graphs below have in the intervals shown.

a)



2

b)



3

The second derivative

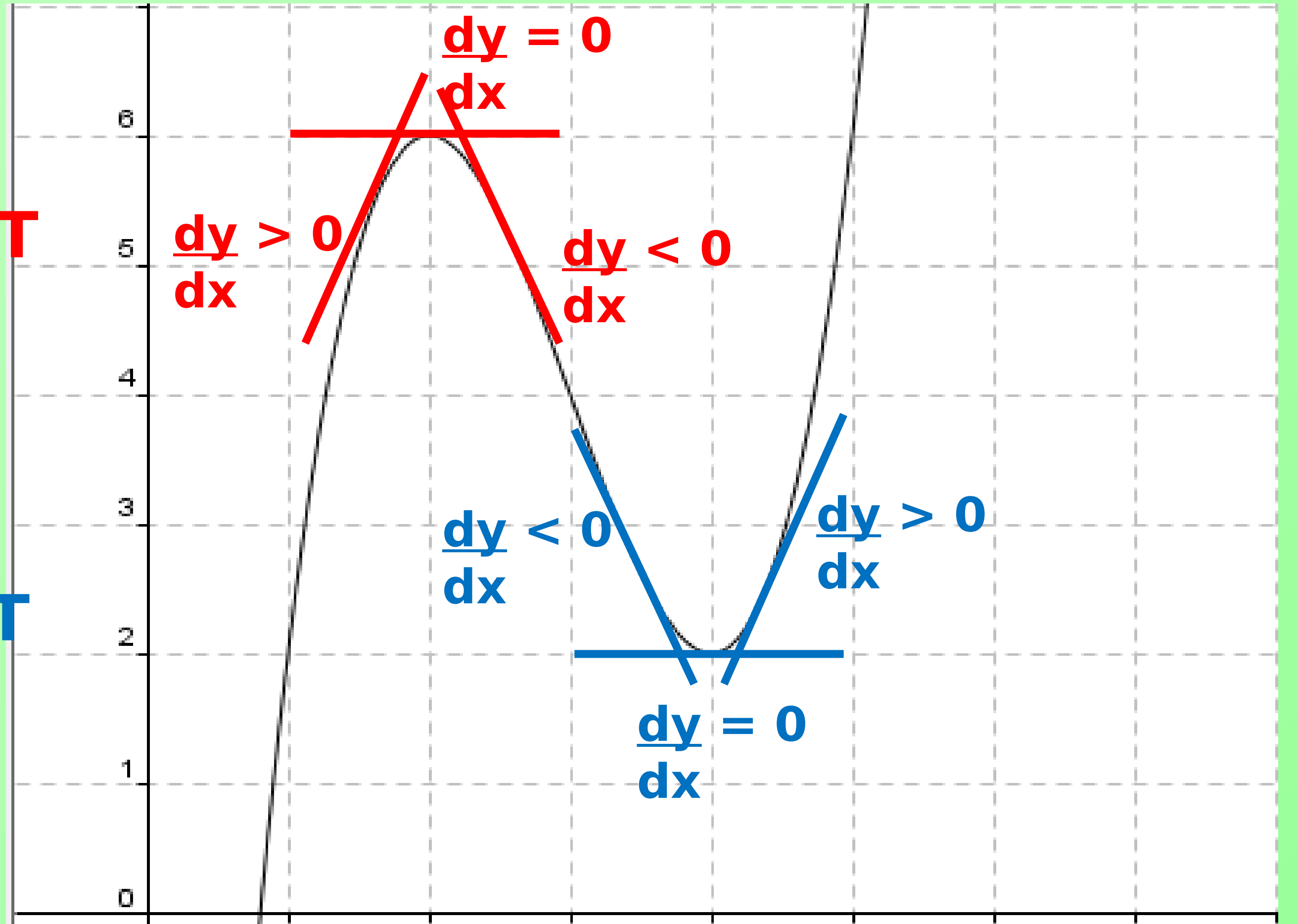
We can use the second derivative to tell us the nature of a stationary point as it tells us ***the rate of change of the gradient***. We can work out if the stationary point is a maximum or a minimum or possibly a po

t MAXIMUM POINT

$$\frac{dy}{dx}=0 \qquad \frac{d^2y}{dx^2}<0$$

t MINIMUM POINT

$$\frac{dy}{dx}=0 \qquad \frac{d^2y}{dx^2}>0$$



4.5 Turning Points

Example 4

Find the stationary points on the curve and determine their nature

t a stationary point,

At :

is a minimum.

At :

is a maximum.

r

tationary points at (1,3) and (-1,7)

4.5 Turning Points

Example 5

Use calculus to work out the coordinates of the stationary point on the curve and determine its nature.

At a stationary point,

(no negative root as)

Stationary point at (1,

4.5 Turning Points

Example 5

Use calculus to work out the coordinates of the stationary point on the curve and determine its nature.

$$\frac{d^2y}{dx^2} = 2x^{-3} = \frac{2}{x^3}$$

At

is a minimum

4.5 Turning Points

You try: Example 6

The curve with equation $y = x^4 - 32x + 5$ has a single stationary point, M

Find the coordinates of M and determine its nature.

Fully justify your answer.

At a stationary point,

At
the stationary point at is a minimum

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4.5 Turning Points

Example 7a

a The curve with equation $y = \frac{1}{x} + 27x^3$ has stationary points at $x = \pm a$. Find the value of a.

b Sketch the graph of $y = \frac{1}{x} + 27x^3$.

At a stationary point,

So

4.5 Turning Points

Example 7b

a The curve with equation $y = \dfrac{1}{x} + 27x^3$ has stationary points at $x = \pm a$. Find the value of a.

b Sketch the graph of $y = \dfrac{1}{x} + 27x^3$.

To sketch the graph we need the coordinates and nature of the turning points:

So the curve has a local minimum at (1/3, 4)

4.5 Turning Points

Example 7b

a The curve with equation $y = \frac{1}{x} + 27x^3$ has stationary points at $x = \pm a$. Find the value of a.

b Sketch the graph of $y = \frac{1}{x} + 27x^3$.

To sketch the graph we need the coordinates and nature of the turning points:

o the curve has a local maximum at (-1/3, -4)

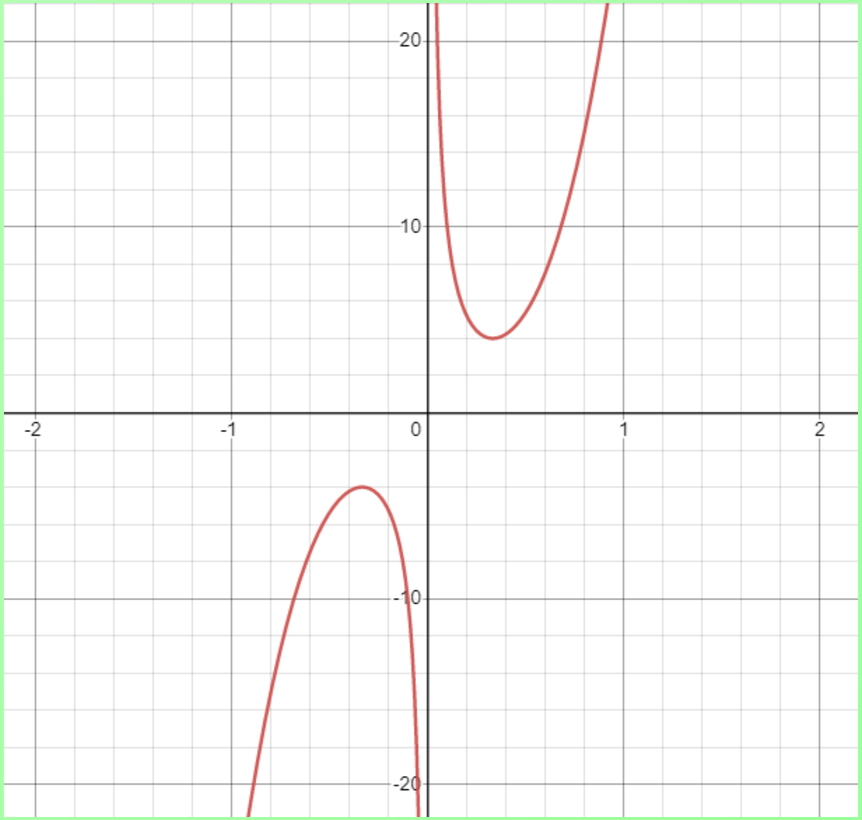
4.5 Turning Points

Example 7b

a The curve with equation $y = \frac{1}{x} + 27x^3$ has stationary points at $x = \pm a$. Find the value of a.

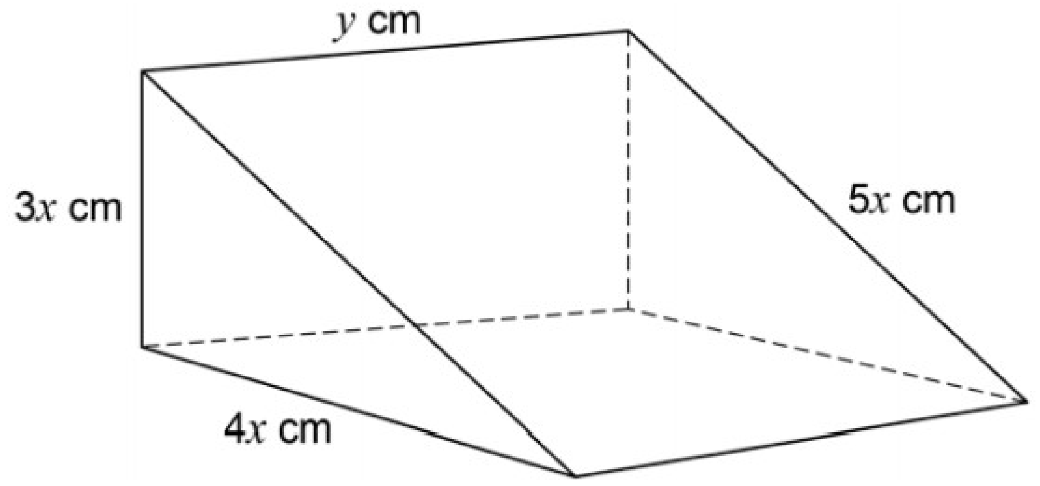
b Sketch the graph of $y = \frac{1}{x} + 27x^3$.

There is also an asymptote at x=0 as we cannot divide by 0.



The diagram shows a block of wood in the shape of a prism with a triangular cross-section. The end faces are right-angled triangles with sides of lengths $3x$ cm, $4x$ cm and $5x$ cm, as shown in the diagram.

Extension Question



The total surface area of the five faces is 144 cm^2.

(a) Show that the volume of the block, V cm^3, is given by $V = 72x - 6x^3$

(b) Show that V has a stationary value when $x = 2$ and determine whether it is a maximum or a minimum.

Fully justify your answer.

Maximum

$$f(x) = ax^2 + bx + c, \; x \in \mathbb{R},$$

where a, b and c are constants.

It is given that $f(x)$ passes through the points $(2,3)$ and $(-1,9)$, and the curve has a stationary point where $x = 1$.

Determine the value of a, b and c ,

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

T.P. $x = 1 \Rightarrow 0 = 2a + b$

$\boxed{b = -2a}$

Now

$$\left.\begin{array}{l}(2,3) \Rightarrow 3 = a \times 2^2 + b \times 2 + c \\ (-1,9) \Rightarrow 9 = a \times (-1)^2 + b(-1) + c\end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{l}3 = 4a + 2b + c \\ 9 = a - b + c\end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{l}3 = 4a + 2(-2a) + c \\ 9 = a - (-2a) + c\end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{l}3 = c \\ 9 = 3a + c\end{array}\right\}$$

$$\left\{\begin{array}{l}c = 3 \\ 9 = 3a + 3\end{array}\right.$$

Hence

$c = 3$ $\quad a = 2$ $\quad b = -2$

The curve C has equation

$$y = x^3 + ax^2 + bx - 10,$$

where a and b are constants.

The curve has two stationary points P and Q.

Given the coordinates of P are $(-1,-5)$, find the coordinates of Q and use $\dfrac{d^2y}{dx^2}$ to determine its nature.

$y = x^3 + ax^2 + bx - 10$

$\dfrac{dy}{dx} = 3x^2 + 2ax + b$

- $(-1,-5) \Rightarrow -5 = -1 + a - b - 10$
- $\left.\dfrac{dy}{dx}\right|_{x=-1} = 0 \Rightarrow 0 = 3 - 2a + b$

$\Rightarrow$

Hence: $a - b = 6$, $-2a + b = -3$ Add $-a = 3$

$\boxed{a = -3}$ & $\boxed{b = -9}$

$\therefore \dfrac{dy}{dx} = 3x^2 - 6x - 9$

$\dfrac{d^2y}{dx^2} = 6x - 6$

Now $3x^2 - 6x - 9 = 0$

$x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0$

$x = -1 \leftarrow P$

$x = 3 \leftarrow Q$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=3} = 6 \times 3 - 6 = 12 > 0$

$y = 3^3 - 3(3)^2 - 9(3) - 10$

$y = \cancel{27} - \cancel{27} - 27 - 10$

$y = -37$

$\therefore Q(3,-37)$

& IT IS + MIN